Generalised G_2-manifolds

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Introduction

Classically, a special geometry over an n-manifold M^n is described in terms of G-structures by a reduction of the canonical $GL(n)$-principal fibre bundle to a proper subgroup $G \leqslant GL(n)$. However, there are other canonical choices. As we will explain below, mathematically [7] as well as physically [5], [9] it is natural to consider geometries associated with the **Narain group**

$$SO(n, n).$$

Basic scenario

To that effect we consider an n-manifold M together with the natural pseudo-Riemannian vector bundle of signature (n, n)

$$T \oplus T^*, \quad (x, \xi) = -\frac{1}{2}\xi(x),$$

and endow $T \oplus T^*$ with its natural orientation. We obtain thus a reduction

$$SO(T \oplus T^*) = SO(n, n) \hookrightarrow GL(T \oplus T^*).$$

Note that

$$GL(T) \leqslant SO(T \oplus T^*)$$

and as a $GL(T)$-space, the Lie algebra $\mathfrak{so}(T \oplus T^*)$ decomposes as

$$\mathfrak{so}(T \oplus T^*) = \operatorname{End} T \oplus \Lambda^2 T^* \oplus \Lambda^2 T.$$

In particular, we get an action of 2-forms $b \in \Lambda^2 T^*$ (to which we refer as "B-fields") by exponentiation to $SO(n,n)$, that is

$$e^b(x + \xi) = x \oplus \xi + x \llcorner b.$$

Spinors

Consider the action

$$(x + \xi) \bullet \rho = x \llcorner \rho + \xi \wedge \rho$$

of $T \oplus T^*$ on Λ^* which defines a Clifford multiplication. Consequently, this induces an isomorphism

$$\mathit{Cliff}(T \oplus T^*) = \operatorname{End}(\Lambda^*)$$

and the spin representations are

$$S^{\pm} = \Lambda^{ev,od}.$$

Morally, we can therefore say that

differential forms = spinors for $\mathbf{T \oplus T^*}$.

The action of a B-field is given by

$$e^b \bullet \rho = (1 + b + \frac{1}{2} b \wedge b + \ldots) \wedge \rho.$$

There is also a $Spin(n,n)$-invariant bilinear form

$$q(\alpha, \beta) = (\alpha \wedge \sigma(\beta))_n,$$

obtained by projection on the top degree n, where σ is the Clifford algebra involution defined on an element α^p of degree p by

$$\sigma(\alpha^p) = \begin{cases} -1, & p \equiv 1, 2 \bmod 4 \\ 1, & p \equiv 0, 3 \bmod 4 \end{cases}$$

Generalised exceptional structures

Next we want to discuss "special" generalised G-structures, that is topological reductions to $G \hookrightarrow SO(n,n)$. Assume we are given a G_2-structure on M^7, i.e. we have a generic 3-form with induced metric g. Then the forms

$$\rho = 1 - \star\varphi \in \Lambda^{ev}, \ \hat{\rho} = -\varphi + \mathrm{vol}_g \in \Lambda^{od}$$

define $Spin(7,7)$-spinors which are stabilised by $G_2 \times G_2$.

Similarly, if we consider a $Spin(7)$-structure on M^8 with associated 4-form Ω, the even form

$$\rho = 1 - \Omega + vol_g = \star\sigma(\rho)$$

is stabilised by $Spin(7) \times Spin(7)$.

In the same vein, there is also a natural notion of a generalised $SU(3)$-geometry [9]. Here, an $SU(3)$-structure achieves a reduction to $SU(3) \times SU(3)$ inside $Spin(6,6)$ through the even and odd forms $\mathrm{Re}(\Omega)$, $\mathrm{Re}(e^{i\omega})$ (or equivalently $\mathrm{Im}(\Omega)$, $\mathrm{Im}(e^{i\omega})$), where Ω denotes the holomorphic volume form and ω the Kähler form.

In the sequel we shall focus on $n = 7$, i.e. the generalised G_2-case. Similar statements hold for $n = 6$ (generalised $SU(3)$-structures) and $n = 8$ (generalised $Spin(7)$-structures). So assume we are given a $G_2 \times G_2$-invariant spinor $\rho \in \Lambda^{ev}$. What does the reduction to

$$G_2 \times G_2 \hookrightarrow Spin(7,7) \to SO(7,7)$$

imply? Firstly,

$$G_2 \times G_2 \leqslant SO(7) \times SO(7),$$

that is we get a **generalised metric**. This means that we have an orthogonal splitting

$$T \oplus T^* = V_+ \oplus V_-$$

into a positive and negative definite subspace V_+ and V_-. Figure 1 suggests how to characterise a metric splitting algebraically. If we think of the coordinate axes T and T^* as a lightcone, choosing a subgroup conjugate to $SO(n) \times SO(n)$ inside $SO(n,n)$ boils down to the choice of an oriented spacelike V_+ and an oriented timelike orthogonal component V_-. Interpreting V_+ as the graph of a linear map $P_+ : T \to T^*$ yields a metric g and a 2-form b as the symmetric and the skew part of the dual $P_+ \in T^* \otimes T^*$. Indeed we have

$$g(t,t) = (t, P_+t) = (t \oplus P_+t, t \oplus P_+t)/2 > 0$$

so that g is positive definite. As V_+ and V_- are orthogonal taking V_- instead of V_+ yields the same 2-form b but the metric $-g$. Conversely, assume we are given a

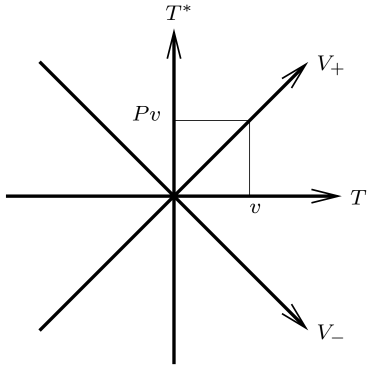

Figure 1: Metric splitting of $T \oplus T^*$

metric g and a 2-form b on T. If we transform the diagonal $D_\pm = \{t \oplus \mp t \llcorner g \mid t \in T\}$ by $\exp(b)$, we obtain a splitting $V_+ \oplus V_-$ inducing g and b.

Now a further reduction to $G_2 \times G_2$ yields a G_2-structure on V_+ and V_- which can be pulled back to give **two** G_2-structures on T. This means that in addition to a generalised metric, we have two spinors in the spin module Δ associated with $Spin(7)$ whose stabiliser we denote by $G_{2\pm}$. The picture to bear in mind is then this. On the manifold, we have an orthonormal frame bundle $P_{SO(n)}$ and reductions to two (possibly non-equivalent) G_2-bundles $P_{G_{2\pm}}$.

Now the spinors stabilised by $G_{2\pm}$ are unique up to a scale, while ρ – the $G_2 \times G_2$ invariant form, is also only determined up to a scale. Hence, we have an additional degree of freedom – the **dilaton** function f. So the data induced by ρ is

- a metric g
- a B-field b
- two unit spinors $\psi_\pm$
- a dilaton f

For the converse we can use the metric to set up the standard identification

$$\Delta \otimes \Delta \cong \Lambda^{ev,od},$$

that is we consider the tensor product of spinors $\varphi \otimes \psi$ as a **form**. We denote its even and odd part by $(\varphi \otimes \psi)^{ev,od}$ respectively. Twisting with the B-field b yields a form

$$(\varphi \otimes \psi)_b^{ev,od} = e^b \wedge (\varphi \otimes \psi)^{ev,od}.$$

Now regard $\Delta \otimes \Delta$ and Λ^* as a $G_2 \times G_2$-**module**.

Theorem 1. *Any $G_2 \times G_2$-invariant form in $\Lambda^{ev,od}$ can be uniquely written as*

$$\rho = e^{-f}(\psi_+ \otimes \psi_-)_b^{ev,od}.$$

Moreover,

$$(\psi_+ \otimes \psi_-)_b^{ev,od} = \square_{g,b}(\psi_+ \otimes \psi_-)_b^{od,ev}$$

where $\square_{g,b} : \Lambda^{ev,od} \to \Lambda^{od,ev}$ is the generalised Hodge operator

$$\square_{g,b}\rho = e^b \wedge \star(e^b \wedge \sigma(\rho)).$$

Definition 1. *A* **topological generalised G_2-structure** *is a pair (M^7, ρ) where ρ is an even or odd form representing a $G_2 \times G_2$-invariant spinor. Equivalently, such a structure is defined by the data $(M^7, g, b, \psi_+, \psi_-, f)$.*

In particular, any classical G_2-manifold (M^7, g, ψ) with $\psi \in \Delta$ induces a generalised G_2-manifold via $(M^7, g, b = 0, \psi_+ = \psi, \psi_- = \psi, f = 0)$. Generalised G_2-structures with $\psi_+ = \psi_- = \psi$ are said to be **straight**. They arise as the B-field transform of a classical G_2-structure.

The variational problem

In the remainder of this talk, we want to discuss **integrable** generalised G_2-structures. But what is a "good" or "natural" integrability condition? In the classical case, the most general condition to ask for is the **flatness** of the underlying G-structure P_G, which measures to which extent the local frames induced by a coordinate system fail to be sections of P_G. The first-order obstruction to flatness is the **intrinsic torsion** of P_G and the usual integrability conditions one imposes are characterised by the vanishing of some or all components of the intrinsic torsion. For example, a **geometrical** G_2-structure (i.e. the holonomy reduces to G_2) is a topological reduction to G_2 with vanishing intrinsic torsion, while a **co-calibrated** G_2-structure allows non-trivial components.

Now geometrical G_2-structures can also be characterised as a critical point of Hitchin's variational principle [6]. More generally, with a **stable form** ρ, i.e. ρ lies in an open orbit under some suitable group action, we can associate an invariant companion form $\hat{\rho}$. Together with ρ it gives rise to a volume form which over a compact manifold can be integrated. Since the orbit is open, this functional can be differentiated. In particular, we can consider its variation over a fixed cohomology class. If ρ is closed, then ρ defines a critical point if and only if $\hat{\rho}$ is closed. For example, the generic

3-form φ defines a stable form with $\hat{\varphi} = \star\varphi$ and a critical point requires φ to be closed an co-closed.

Analogously to the classical G_2-case, the orbit of a $G_2 \times G_2$-invariant form in $\Lambda^{ev,od}$ under the action of $\mathbb{R}_{>0} \times Spin(7,7)$ is open

$$\dim \mathbb{R}_{>0} \times Spin(7,7) - \dim G_2 \times G_2 = 92 - 28 = \dim \Lambda^{ev,od},$$

that is the form is **stable**. This open orbit is effectively parametrised by the data above, i.e.

$$\begin{array}{c} f:1 \quad b:21 \quad g:28 \quad \psi_+:7 \quad \psi_+:7 \\ 1+21+28+7+7 = 64 = \dim \Lambda^{ev,od}. \end{array}$$

Here the $\wedge$-operation is just

$$\hat{\rho} = \Box_{g,b}\rho,$$

where g and b is induced by ρ. Now we can define a volume form by

$$\phi(\rho) = q(\rho, \hat{\rho})$$

and consider various variational problems. The critical points of the **unconstrained** problem are precisely those with

$$d\rho = 0, \quad d\Box\rho = 0,$$

and we call such a generalised G_2-structure **integrable**. A **constrained** version of the variational principle yields the equations

$$d\rho = \lambda\hat{\rho},$$

where λ is a real constant. According to the parity of ρ, we call such a generalised G_2-structures **weakly integrable of even or odd type**.

Example: Any **geometrical** G_2-manifold defines an integrable generalised G_2-manifold, since

$$\rho = 1 - \star\varphi, \quad \hat{\rho} = -\varphi + vol$$

are both closed forms. On the other hand one can show that any weakly integrable G_2-manifold which is straight is actually integrable. In this sense, there is no classical counterpart to the notion of weak integrability.

Spinorial solution of the variational problem

Theorem 2. *A generalised G_2-structure (M^7, ρ) is (weakly) integrable (of even or odd type) if and only if for any vector field X $e^{-f}(\psi_+ \otimes \psi_-)_b = \rho + \Box_\rho \rho$ satisfies*

$$\begin{aligned} &\nabla_X \psi_\pm \pm \tfrac{1}{8} X \llcorner db \cdot \psi_+ = 0 \\ &(df \pm \tfrac{1}{4} db \pm \lambda) \cdot \psi_\pm = 0 \text{ (even type)} \\ &(df \pm \tfrac{1}{4} db + \lambda) \cdot \psi_\pm = 0 \text{ (odd type)} \end{aligned}$$

We refer to the equation involving the covariant derivative of the spinor as the **generalised Killing equation** and to the equation involving the differential of f as the **dilaton equation**. The generalised Killing equation basically states that we have two metric connections $\nabla^\pm$ preserving the underlying $G_{2\pm}$-structures whose torsion (as it is to be defined in the next section) is **skew**. The dilaton equation then serves to identify the components of the torsion with respect to the decomposition into irreducible $G_{2\pm}$-modules with the additional data df and λ. The generalised Killing and the dilaton equation occur in physics in bosonic type IIA/B supergravity and superstring theory [4], [9].

Proof: (sketch) It is well-known that the twisted Dirac operator $\mathcal{D}$ transforms under the standard isomorphism $\varphi \otimes \psi \in \Lambda^*$ into $d + d^*$. For the twisted isomorphism $(\varphi \otimes \psi)_b$ we find

$$\mathcal{D}(\varphi \otimes \psi)_b = d(\varphi \otimes \psi)_b + d^\Box(\varphi \otimes \psi)_b + \frac{1}{2} e^{b/2} \wedge (db \llcorner (\varphi \otimes \psi) - db \wedge (\varphi \otimes \psi)),$$

where $d^\Box \alpha^p = (-1)^{n(p+1)+1} \Box d \Box \alpha^p$. Using this plus the usual spinor algebra rules yields the result. ■

Remark: More generally, we can replace db by a closed 3-form. In terms of the variational problem this corresponds to the variation over a twisted cohomology class rather than a usual de Rham cohomology class.

Geometrical properties

The theorem asserts that we have two connections ∇^+ and ∇^- with **skew-symmetric torsion** $\pm db$, each of which preserving the $G_{2\pm}$-structure defined by the spinors $\psi_\pm$. This kind of connections was studied in a series of papers by Friedrich and Ivanov [2], [8], [3]. In particular, we have

$$db = \mp e^{-2f} \star (d e^{-2f} \varphi_\pm)$$

which implies for a compact M that

$$\int_M e^{-2f} db \wedge \star db = \mp \int_M db \wedge d(e^{-2f}\varphi_\pm) = 0.$$

Consequently, any integrable generalised $G_2 \times G_2$ structure on a compact manifold consists of two geometrical G_2-structures (i.e. there are only "trivial" solutions for the variational problem). However, there are compact examples of weakly integrable structures.

What about homogenous structures?

Proposition 3. *The Ricci-tensor of an integrable generalised G_2 structure is given by*

$$\mathrm{Ric}(X,Y) = -\frac{7}{2}H^f(X,Y) + \frac{1}{4}g(X \llcorner db, Y \llcorner db),$$

where $H^f(X,Y) = X.Y.F - \nabla_X Y.f$ is the Hessian of the dilaton f. The structure is Ricci-flat if and only if $df = 0$ (implying $db = 0$). In particular, there are no non-trivial homogeneous examples.

T-duality

For the construction of examples, the form approach is most useful. Here we will discuss the device of T-duality which comes from string theory where it interchanges type IIA with type IIB. It can be applied in the situation where we are given an S^1-fibre bundle P with connection form θ [1]. T-duality associates with this data another S^1-principal fibre bundle P^T over the same base, but with a different connection 1-form θ^T. The T-dual of the spinor

$$\rho = \theta \wedge \rho_0 + \rho_1$$

of a generalised structure is defined to be

$$\rho^T = \theta^T \wedge \rho_1 + \rho_0.$$

In particular, ρ is closed if and only if ρ^T is closed. Moreover, $(\Box\rho)^T = \Box_{\rho^T}\rho^T$ and hence T-duality preserves strong integrability. It also exchanges parity and preserves weak integrability so that in particular, a weakly integrable generalised G_2-structure of even type becomes a weakly integrable structure of odd type.

We then construct non-trivial (integrable) examples by T-dualising a straight geometrical G_2-structure over an S^1-bundle (local examples hereof exist in abundance). Schematically twisted S^1 structure, but straight $G_2 \times G_2$ structure

which we replace by P^T with a flat connection form θ^T, i.e.

trivial S^1 structure, but non-straight $G_2 \times G_2$ structure.

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