Notes on Mirror Symmetry

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1 Introduction

The purpose of these notes is for me to recollect what I have been trying to learn about the subject of mirror symmetry and to explain it to others. I describe, with as little technical language as I can, some aspects of mirror symmetry. In particular I discuss Calabi-Yau manifolds and their Hodge diamond, deformation theory of complex structures and the Bogomolov-Tian-Todorov theorem. I will outline the concept of Large Complex Structure Limit point in the case of 1-dimensional complex moduli, discuss periods, monodromy and the computation of the Yukawa coupling which provides a formula for the number of rational curves on the mirror manifold. As inevitable, I make extensive use of classical theorems of Kählerian geometry such as those found in the first two chapters of Griffiths and Harris [2]. These notes are largely based on Part II of the book [3], which I recommend to anyone wanting to learn about the subject for the first time. Another very good book is [1], which has a lot of interesting material, but starts at a higher level and the emphasis is more on algebraic geometry, especially toric. The survey articles of David Morrison, such as [6] are also quoted as a good reference.

2 Calabi-Yau's and their Hodge numbers

Mirror symmetry is a, so far, rather mysterious construction relating apparently unrelated families of some special kinds of Kähler manifolds called Calabi-Yau manifolds. Let X be an n-dimensional complex manifold and

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$K_X = \Lambda^n T^* X^{(1,0)}$ the complex line bundle of forms of type $(n,0)$, also called the canonical bundle of X.

Definition 1 *An n-dimensional complex manifold X is Calabi-Yau if it is Kählerian, $\pi_1(X) = 0$ and $c_1(K_X) = 0$, where c_1 is the first Chern class.*

Proposition 1 *The line bundle K_X of a compact Calabi-Yau manifold X is holomorphically trivial, i.e. X admits a nowhere vanishing global holomorphic $(n,0)$-form Ω.*

Proof. The line bundle K_X is obviously holomorphic. It is known that holomorphic line bundles are classified by the so called Picard group $H^1(X, \mathcal{O}^*)$, where $\mathcal{O}^*$ is the sheaf of nowhere zero holomorphic functions (cfr. [2] pgg. 132-133). The 0 in $H^1(X, \mathcal{O}^*)$ is the class of holomorphically trivial line bundles. Remember the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0,$$

where $\mathcal{O}$ is the sheaf of holomorphic functions, the second arrow is inclusion and the the third is $f \mapsto e^{2\pi i f}$. Corresponding to it is the long exact sequence of cohomology, one part of which is:

$$0 = H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}). \tag{1}$$

The fact that $H^1(X, \mathcal{O}) = 0$ follows from Dolbeaut's theorem ([2], pg.45), saying that $H^1(X, \mathcal{O}) \cong H^{0,1}(X, \mathbb{C})$, then from Hodge's decomposition (which holds since X is Kählerian, [2] pg.116):

$$H^k(X, \mathbb{C}) = \bigoplus_{l=1}^{k} H^{k-l,l}(X, \mathbb{C}),$$

and finally from the assumption that X is simply connected. The last arrow of (1) now tells us that c_1 is injective, i.e. that a line bundle with vanishing c_1 is holomorphically trivial. $\square$

Let $h^{p,q}$ denote the (p,q)-Hodge number of X, i.e. the dimension of $H^{p,q}(X, \mathbb{C})$. Remember that from Hodge theory, when X is Kähler, we have that:

$$\overline{H^{p,q}(X, \mathbb{C})} = H^{q,p}(X, \mathbb{C}),$$

and therefore:

$$h^{p,q} = h^{q,p}.$$

Instead from Serre duality we have:

$$H^{p,q}(X,\mathbb{C}) \cong H^{n-p,n-q}(X,\mathbb{C}),$$

and therefore:

$$h^{p,q} = h^{n-p,n-q}.$$

When X is Calabi-Yau, the previous Proposition and Dolbeaut's theorem imply that $h^{n,0} = h^{0,n} = 1$ and $h^{1,0} = h^{0,1} = 0$. Moreover observe that

$$H^{n,1}(X,\mathbb{C}) \cong H^1(X,\Omega_X^n) \cong H^1(X,\mathcal{O}) = 0,$$

where the first isomorphism is Dolbeaut and the second follows from the definition of Calabi-Yau. From Serre duality we also get

$$h^{0,n-1} = h^{n-1,0} = 0.$$

In particular we obtain

Proposition 2 *The Hogde diamond of a Calabi-Yau* 3*-fold* X *looks like:*

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{1,2} & & h^{1,2} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

3 Examples

Let's now look for interesting examples. We start from dimension 1.

3.1 Dimension 1.

All complex curves are Kählerian, but the only simply connected one is $\mathbb{P}^1$, whose canonical bundle is not trivial. Therefore, strictly speaking there are no 1 dimensional Calabi-Yau manifolds, but if we just require X to have holomorphically trivial canonical bundle, i.e. $h^{1,0} = 1$, then we must have $b^1(X) = 2$, i.e. X is topologically a torus. In fact let $\Lambda \subseteq \mathbb{C}$ be a lattice of maximal rank, i.e. $\Lambda = \mathrm{span}_{\mathbb{Z}}\langle \tau_1, \tau_2 \rangle$, where τ_1 and τ_2 are two $\mathbb{R}$-independent complex numbers. Then $X = \mathbb{C}/\Lambda$ is a genus 1 curve whose canonical bundle is trivialized by the form $\Omega = dz$, where z is a coordinate on $\mathbb{C}$. It turns out that by some simple symmetries any such X is biholomorphic to one where $\tau_1 = 1$ and $\tau_2 \in \mathcal{H}$, where $\mathcal{H}$ is the upper half plane in $\mathbb{C}$.

Exercise. Let $p \in \mathbb{C}[x_0, x_1, x_3]$ be a homogeneous polynomial of degree 3. Prove that, for generic p, $X = \{p = 0\} \subset \mathbb{P}^2$ is a smooth genus 1 curve. Moreover show that the one form

$$\Omega = \frac{dx_1}{\partial p / \partial x_2},$$

seen as a form with poles on the affine chart $\{x_0 \neq 0\}$, restricts and extends to a well defined, nowhere vanishing holomorphic 1-form on X. Notice how degree 3 of p is the only case when this happens. □

3.2 Hypersurfaces

We now look for Calabi-Yau manifolds among hypersurfaces of known complex manifolds, such as $\mathbb{P}^{n+1}$. A useful tool is the adjunction formula. Given a hypersurface X of some Kähler manifold M, let N_X denote the normal bundle of X inside M. It is obviously a holomorphic line bundle of X. We have:

Theorem 1 (The adjunction formula) *Given X, a hypersurface of an $n+1$-dimensional complex manifold M, we have:*

$$K_X = K_{M_{|X}} \otimes N_X.$$

Proof. Roughly, if Ω is a local section of K_M, i.e. a holomorphic $n+1$ form around a point in X and ν is a holomorphic section of N_X, then the contraction $\iota_\nu \Omega_{|X}$ is a holomorphic n form on X. This gives the above identification. □

Now suppose X is a hypersurface of degree d in $\mathbb{P}^{n+1}$, i.e. X is the zero locus of some degree d homogeneous polynomial. When is $c_1(K_X) = 0$? One practical way to compute c_1 of some holomorphic line bundle L is to look for a meromorphic section σ of L. If σ has poles of degree q_k along hypersurfaces P_k and zeroes of degree r_l along hypersurfaces Z_l then:

$$c_1(L) = PD(\sum_l r_l Z_l - \sum_k q_k P_k),$$

where PD denotes the Poincare dual in homology. For a more rigorous treatment of these facts look in Griffiths and Harris, Section 1.1, Divisors and Line Bundles.

So for example

$$c_1(K_{\mathbb{P}^{n+1}}) = -(n+2)PD(H),$$

where H is the class of a hyperplane. You can see this by trying to define a holomorphic $n+1$-form Ω on $\mathbb{P}^{n+1}$ by taking $\Omega = dx_1 \wedge \ldots \wedge dx_{n+1}$ on the affine chart $U_0 = \{x_0 \neq 0\}$. When you change chart, say on U_1, you find that Ω has a pole of degree $n+2$ on $H_0 = \{x_0 = 0\}$.

On the other hand if X is a hypersurface of M, sections of the normal bundle N_X correspond to infinitesimal deformations of X inside M, therefore they will vanish, intuitively, along a hypersurface of X reppresenting $X \cdot X$, the self intersection class.

In particular if X is of degree d in $\mathbb{P}^{n+1}$, then:

$$c_1(N_X) = PD(X \cap X) = PD(dH \cap X),$$

since X is homologous to the class dH. We may now apply the adjunction formula:

$$c_1(K_X) = c_1(K_{\mathbb{P}^{n+1}})_{|X} + c_1(N_X) = PD((d-n-2)H \cap X).$$

We may then conclude:

Proposition 3 *A smooth hypersurface X of degree d in $\mathbb{P}^{n+1}$ is such that $c_1(K_X) = 0$ if and only if $d = n+2$.*

So low dimensional examples of such hypersurfaces are quartics in $\mathbb{P}^3$, also known as K3 surfaces and quintics in $\mathbb{P}^4$. Notice also that the 1 dimensional case of cubics in the previous section is a particular case of this result. In the following exercise you can verify the above proposition directly, also obtaining an explicit holomorphic n-form.

Exercise Let X be a smooth, degree $d = n+2$ hypersurface in $\mathbb{P}^{n+1}$ and let f be the homogeneous polynomial defining X. On an affine chart for $\mathbb{P}^{n+1}$, say $U_0 = \{x_0 \neq 0\}$, define a holomorphic n-form (with poles):

$$\Omega_f = \frac{dx_2 \wedge \ldots \wedge dx_{n+1}}{\partial f / \partial x_1}.$$

Prove that it restricts and then extends to a well defined nowhere vanishing holomorphic n-form on X. $\square$

We have not yet shown that our examples are simply connected. We will only show:

Proposition 4 *Given any smooth hypersurface X of $\mathbb{P}^{n+1}$ with $n \geq 2$, we have*

$$H^q(X, \mathbb{Q}) \cong H^q(\mathbb{P}^{n+1}, \mathbb{Q}), \tag{2}$$

when $q \leq n-1$.

We apply Lefschetz Theorem on hyperplane sections (cfr. [2, pg. 156]) which says the following. Let M be *any* smooth $n+1$-dimensional submanifold of $\mathbb{P}^N$ for some N and let $V = H \cap M$, where H is a hyperplane. For sufficiently general H, V will be a smooth manifold, called a hyperplane section of M. Lefschetz's theorem says that the map

$$H^q(M, \mathbb{Q}) \to H^q(V, \mathbb{Q})$$

given by restriction, is an isomorphism for $q \leq n-1$ and injective for $q = n$. Now we consider X to be a degree d hypersurface in $\mathbb{P}^{n+1}$. Take the d-tuple Veronese embedding Φ of $\mathbb{P}^{n+1}$ into $\mathbb{P}^N$, where $N = \begin{pmatrix} n+1-d \\ n+1 \end{pmatrix} - 1$. It is defined by $[x_0 : \ldots : x_{n+1}] \mapsto [\mu_0 : \ldots : \mu_N]$ where the μ_k's range among all possible degree d monomials in the x_j variables. Let $M = \Phi(\mathbb{P}^{n+1})$. We can easily see that for a suitable hyperplane H of $\mathbb{P}^N$, we have $\Phi(X) = M \cap V$. Lefschetz's theorem applied to such M and V and for $q \leq n-1$, gives 2. □

3.3 The cohomology of the quintic

We will now compute the Hodge numbers of a smooth quintic X in $\mathbb{P}^4$, which, as we just saw, is Calabi-Yau. Proposition 2 tells us that we only need to find $h^{1,1}$ and $h^{2,1}$, while from Proposition 2 we already obtain that $h^{1,1} = 1$. Therefore we can obtain $h^{2,1}$ from the Euler characteristic of X through the formula:

$$\mathcal{X}(X) = 2 - 2h^{2,1}.$$

The generalized Gauss-Bonnet Theorem tells us that, on an n dimensional complex manifold:

$$\mathcal{X}(X) = \int_X c_n(TX).$$

To apply it to the quintic we need to compute $c_3(X)$. Let $L \to \mathbb{P}^4$ be the tautological line bundle of $\mathbb{P}^4$ defined by:

$$L = \{(p, \ell) \in \mathbb{C}^5 \times \mathbb{P}^4 | p \in \ell\},$$

and let L^* be its dual. The sheaf of holomorphic sections of L and L^* is often denoted $\mathcal{O}_{\mathbb{P}^4}(-1)$ and $\mathcal{O}_{\mathbb{P}^4}(1)$ respectively. We have $c_1(L^*) = PD(H)$. We need the following two facts:

$$T\mathbb{P}^4 \oplus \mathbb{C} \cong (L^*)^{\oplus 5}$$

$$TX \oplus N_X \cong T\mathbb{P}^4_{|X}$$

A proof of the first fact can be found in [4, pg. 190] or can be checked with some thought. The second fact is obvious. Let $c = \sum c_j$ be the total Chern class. From the first ismorphism we get:

$$c(T\mathbb{P}^4) = (1 + PD(H))^5.$$

From which we get

$$c_j(T\mathbb{P}^4) = \left(\begin{array}{c} 5 \\ j \end{array} \right) PD(H)^j, \quad \text{when } j = 1 \ldots 4$$

Applying c to the second isomorphism we have

$$c(TX)(1 + c_1(N_X)) = \sum_j^3 \left(\begin{array}{c} 5 \\ j \end{array} \right) PD(H^j \cap X).$$

As we already know $c_1(N_X) = PD(5H \cap X)$. Then, by expanding the lefthand side, using $c_1(TX) = 0$ and comparing degrees, we obtain

$$c_2(TX) = 10\, PD(H^2 \cap X)$$

$$c_3(TX) + c_2(TX)\, PD(5H \cap X) = 10\, PD(H^3 \cap X),$$

from which we obtain

$$c_3(TX) = -40\, PD(H^3 \cap X).$$

Integrating we get

$$\int_X c_3(TX) = -40\, H^3 \cap X.$$

Now, in $\mathbb{P}^4$, H^3 is the class reppresented by a line, which generically intersects X in 5 points. Therefore

$$\mathcal{X}(X) = \int_X c_3(TX) = -200,$$

giving us $h^{2,1} = 101$. We summarize the result in

Proposition 5 *The Hogde diamond of a smooth quintic X in $\mathbb{P}^4$ is:*

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

- intersections - other examples

4 The complex moduli

One of the main ingredients in the mirror symmetry construction is the space of deformations of the complex structure of a Calabi-Yau manifold X, the so called complex moduli of X and denoted by $\mathcal{M}_{cx}(X)$. We will first review some facts of the general theory of deformations of complex structures.

4.1 Deforming the complex structure

Let X be any compact complex n-dimensional manifold and denote by J its integrable almost complex structure, i.e. $J \in \Gamma(\mathrm{End}\, TX)$, $J^2 = -\mathrm{Id}$ and $N_J = 0$, where N_J is the Nijenhuis tensor associated to J (cfr. [5, pg. 123]). Clearly the space of diffeomorphisms *Diff*(X) of X acts on (integrable) almost complex structures by $J \mapsto \phi^* J$, where $\phi \in$ *Diff*(X). Ideally we would like understand the space all possible integrable J's on the same underlying differentiable manifold X after quotienting by this action, i.e. we are interested in:

$$\mathcal{M}_{cx}(X) = \frac{\{J \in \Gamma(\mathrm{End}\, TX) | J^2 = \mathrm{Id}, N_J = 0\}}{\mathit{Diff}(X)}$$

Usually such a space is very complicated and very little is known about it. More often, what one can do is understand this space locally around a given complex structure J_0. We can formulate the problem formally with the following definition:

Definition 2 *A local deformation space of a compact complex manifold X consists of the data $(\mathcal{X}, S, p, s_0)$, where X and S are analytic spaces, $p : \mathcal{X} \to S$ is a surjective, proper and flat morphism and $s_0 \in S$ is such that $\mathcal{X}_0 = p^{-1}(S_0)$ is isomorphic to X.*

To clarify a bit the definition, keep in mind that it implies that in the case S is isomorphic to some contractible open set in some $\mathbb{C}^r$ then $\mathcal{X}$ is diffeomorphic to $X \times S$ and p is the projection. Therefore what is varying here is the complex structure J on the same underlying manifold X. The definition says the variation is analytic, i.e. J depends analytically on a parameter $s \in S$. One would like to have the biggest of such deformation spaces, i.e. one that contains all the others and which doesn't contain any trivial deformations. Here is the definition:

Definition 3 *The deformation space $(\mathcal{X}, Def(X), p, s_0)$ is the universal deformation space of X if for any other deformation space $(\mathcal{X}', S, p', s_0')$ of X there exists a unique analytic map $f : S \to Def(X)$ such that $f(s_0') = s_0$ and the deformation space $(f^*(\mathcal{X}), S', f^*(p), s_0')$ is isomorphic to $(\mathcal{X}', S, p', s_0')$, i.e. one has the following commuting diagram:*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\simeq} & f^*(\mathcal{X}) \\ p' \downarrow & & \downarrow f^*(p) \\ S & \xrightarrow{\mathrm{Id}} & S \end{array}$$

Often one denotes by just *Def*(X) the universal deformation space. The existence of the function f says that *Def*(X) contains all possible deformations of X, while uniquenes tells us that *Def*(X) does not contain repetitions. In general *Def*(X) may not exist or may be singular. The theory of deformations of complex structures has been develloped by various people such as Kuranishi, Kodaira and Spencer. They studied obstruction to the existence of *Def*(X) and conditions guaranteeing that *Def*(X) is smooth. Let $\mathcal{T}_X$ denote the sheaf of holomorphic sections of the holomorphic tangent bundle of X. The results are stated in terms of the cohomology of $\mathcal{T}_X$.

Theorem 2 (Kuranishi) *If $H^0(X, \mathcal{T}_X) = 0$ then a universal deformation space exists.*

The condition is saying that X should not admit holomorphic vector fields, which induce a non-discrete space of automorphisms. Moreover we have

Theorem 3 (Kodaira-Spencer) *Tangent vectors to Def(X) at s_0 are naturally identified with classes in $H^1(X, \mathcal{T}_X) = 0$. If also $H^2(X, \mathcal{T}_X) = 0$ then Def(X) can be identified with an open neighbourhood of 0 in $H^1(X, \mathcal{T}_X) = 0$, in particular Def($X$) is smooth.*

Some references to these results are ....

Although we will not prove these theorems, we can at least say how one starts to prove them. Let TX be the real tangent bundle of X. The existence of an almost complex structure J is equivalent to the existence of a smooth splitting

$$TX \otimes_{\mathbb{R}} \mathbb{C} = TX^{(1,0)} \oplus TX^{(0,1)}$$

such that $TX^{(0,1)} = \overline{TX^{(1,0)}}$. In fact $TX^{(1,0)} = \{V - iJV | V \in TX\}$ and when J is integrable $TX^{(0,1)}$ is the holomorphic tangent bundle which is often identified with TX itself. Integrability of J is equivalent to the fact that the Lie bracket of two vector fields of type $(1,0)$ remains of type $(1,0)$. The dual splitting of the cotangent bundle

$$T^*X \otimes_{\mathbb{R}} \mathbb{C} = T^*X^{(1,0)} \oplus T^*X^{(0,1)}$$

is the usual one in terms of forms of type $(1,0)$ and $(0,1)$. Let us now denote by X_0 the manifold X with a specific choice of complex structure J_0 and by X_1 the same X but with a different almost complex structure J_1. We say that X_1 is near X_0 if $TX_1^{(0,1)}$ is the graph, inside $TX \otimes \mathbb{C}$, of a linear map $-\alpha : TX_0^{(0,1)} \to TX_0^{(1,0)}$ (taking the minus sign is just a convention). In fact it can be easily proved that if α is a sufficiently small such map, then the subspace of $TX \otimes \mathbb{C}$ defined by $T_\alpha = \{V - \alpha(V) | V \in TX_0^{(0,1)}\}$, i.e. the graph of $-\alpha$, defines a splitting $TX \otimes \mathbb{C} = T_\alpha \oplus \overline{T_\alpha}$ and therefore an almost complex structure J_1 such that $TX_1^{(0,1)} = T_\alpha$.

Now a linear map $\alpha : TX_0^{(0,1)} \to TX_0^{(1,0)}$ is a section of the bundle $T^*X_0^{(0,1)} \otimes TX_0^{(1,0)}$, i.e. $\alpha \in \Omega^{(0,1)}(X_0, TX_0)$, the space of $(0,1)$-forms with values in the tangent bundle (which we identify with $TX_0^{(1,0)}$). Therefore nearby almost complex structures can be identified with a small neighbourhood of the zero section of $\Omega^{(0,1)}(X_0, TX_0)$. If $z = (z_1, \ldots, z_n)$ is a set of holomorphic local coordinates on X_0, then α as above can be written as:

$$\alpha = a^{hk} \frac{\partial}{\partial z_h} \otimes d\overline{z}_k.$$

When is the almost complex structure defined by α also integrable? To answer this question we first define a pairing:

$$[\cdot, \cdot] : \Omega^{(0,1)}(X_0, TX_0) \otimes \Omega^{(0,1)}(X_0, TX_0) \to \Omega^{(0,2)}(X_0, TX_0).$$

If $\beta = b^{hk} \frac{\partial}{\partial z_h} \otimes d\overline{z}_k$ is another section then we define

$$[\alpha, \beta] = \left[a^{hl} \frac{\partial}{\partial z_h}, b^{hk} \frac{\partial}{\partial z_h} \right] \otimes d\overline{z}_l \wedge d\overline{z}_k,$$

where the bracket on the righthand side is the usual one. One can easily check that it is defined independently of the holomorphic coordinates. We have:

Theorem 4 *Given $\alpha \in \Omega^{(0,1)}(X_0, TX_0)$ sufficiently close to the zero section, the almost complex structure it defines is integrable if and only if it satisfies:*

$$\overline{\partial}\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \tag{3}$$

Proof. By definition, a local basis $s_1, \ldots, s_n$ of $(0,1)$-vectors with respect to the new almost complex structure is given by:

$$s_k = \frac{\partial}{\partial \overline{z}_k} - a^{hk}\frac{\partial}{\partial z_h}.$$

A corresponding basis of (1,0)-forms is:

$$\tau_k = dz_k + a^{kh} d\overline{z}_h$$

The way to prove that they are $(1,0)$-forms is to show that $\tau_k(s_l) = 0$ for all k and l. As mentioned earlier, the almost complex structure is integrable if and only if the Lie bracket of $(0,1)$-vectors is still a $(0,1)$-vector, i.e. if and only if $[s_k, s_l] \in TX_1^{(0,1)}$ for all k and l. This is the same as saying

$$\tau_j([s_k, s_l]) = 0 \qquad \forall j, k, l = 1, \ldots, n$$

By writing it explicitely, this condition is easily seen to coincide with equation 3. $\square$

Now assume that $\alpha(t)$ is an analytic family of nearby integrable complex structures, i.e. depending analytically on a parameter $t \in \mathbb{C}$, such that $\alpha(0) = 0$. Denote by $\dot{\alpha}(t)$ the derivative of $\alpha(t)$ with respect to t. Since $\alpha(t)$ satisfies (3) for all t, taking the derivative of (3) w.r.t. t and evaluating at $t = 0$ we see that

$$\overline{\partial}\dot{\alpha}(0) = 0.$$

This is saying that an infinitesimal deformation of a complex structure defines a cohomology class in $H^{(0,1)}(X, TX_0)$. When is $\dot{\alpha}(0)$ exact? Well, suppose that all $\alpha(t)$'s reppresent the same complex structure, i.e. that there exists a family of diffeomorphisms $F_t : X \to X$ such that $F_t^*(\alpha(t)) = \alpha(0)$. Let

$$V = \frac{\partial F}{\partial t}_{|t=0}.$$

Notice that $V \in TX \otimes \mathbb{C}$. Define V' to be the $(1,0)$ part of V, then one can prove

Proposition 6 *A section $\dot{\alpha}(0)$ as above is the infinitesimal deformation of a trivial family of complex structures if and only if there exists $V' \in \Gamma(TX_0^{(1,0)})$ such that*

$$\dot{\alpha}(0) = -\overline{\partial}V'.$$

Remember that Dolbeaut's theorem tells us that

$$H^{(0,1)}(X, TX_0) = H^1(X, \mathcal{T}_X).$$

Therefore the arguments just outlined illustrate the first part of Theorem 3. The identification of a tangent vector to *Def*(X) with a class of $H^1(X, \mathcal{T}_X)$ is also called the Kodaira-Spencer map. To proceed in the proof of Theorem 3 one expresses $\alpha(t) = \sum_{r=1}^{+\infty} \alpha_r t^r$, where $\alpha_r \in \Omega^{(0,1)}(X_0, TX_0)$. Then tries to solve (**??**) recursively in the $\alpha_r' s$ and checks for convergence. After a lot of work one finds that the only obstruction is the vanishing of $H^2(X, \mathcal{T}_X)$.

4.2 What about Calabi-Yau's?

Our goal is to understand *Def*(X) of a Calabi-Yau manifold X. The first obvious thing to do is to interpret the vector spaces $H^*(X, \mathcal{T}_X)$ and see what we can say about them. Notice that in the Calabi-Yau case, a choice of non-zero holomorphic n-form Ω induces isomorphism

$$I_\Omega : H^{(0,*)}(X, TX) \to H^{(n-1,*)}(X, \mathbb{C}).$$

At the level of sections this isomorphism is as follows. If $\phi \otimes v \in \Omega^{(0,r)}(X, TX)$, where ϕ is the form and v the vector, then:

$$\phi \otimes v \mapsto \iota_v \Omega \wedge \phi.$$

The right-hand side is an $(n-1, r)$-form. One can see that this map is surjective at the level of forms and well defined and bijective at the level of cohomology (e.g. using harmonic representatives from Hodge theory ??). In the 3 dimensional case, from these isomorphisms and Dolbeaut's theorem we get

$$H^0(X, \mathcal{T}_X) = H^{2,0}(X, \mathbb{C})$$
$$H^1(X, \mathcal{T}_X) = H^{2,1}(X, \mathbb{C})$$
$$H^2(X, \mathcal{T}_X) = H^{1,1}(X, \mathbb{C})$$

The first two equalities are good, because they tell us that applying Kuranishi's theorem we do have a universal deformation space *Def*(X) and that we can expect its dimension to be $h^{2,1}$, one of the Hodge numbers we were able to compute for example for the quintic. The last isomorphism instead is bad news, we cannot apply Kodaira-Spencer to Calabi-Yau's to have a smooth deformation space. Nevertheless the existence of a smooth universal deformation space has been proved for Calabi-Yau's:

Theorem 5 (Bogolomov, Tian, Todorov) *Given an dimensional Calabi-Yau manifold X, a smooth deformation space $Def(X)$ of X exists of dimension $h^{n-1,1}$.*

We will not prove this theorem. This gives a nice interpretation of the hodge number $h^{n-1,1}$ as the dimension of the complex moduli of a Calabi-Yau manifold. In particular, in the case of the quintic this number is 101, so quite big.

One last important fact which is worth mentioning is what happens to the Kähler condition as we deform the complex structure. There is a nice stability theorem due to Kodaira-Spencer:

Theorem 6 (Kodaira-Spencer Stability) *Let $(\mathcal{X}, S, p, s_0)$ be a local deformation space of the compact complex manifold $\mathcal{X}_0 = p^{-1}(s_0)$. If $\mathcal{X}_0$ is Kähler, then there exists a neighbourhood U of s_0 such that $\mathcal{X}_s = p^{-1}(s)$ is also Kähler for every $s \in U$.*

There are counter examples due to Hironaka showing that for distant values of the parameter the Kähler condition may be lost. For Calabi-Yau manifolds there are stronger stability results concerning the structure of the Kähler cone, i.e. the set of degree 2 cohomology classes which admit as reppresentative a Kähler form. These results are proved by P.H. Wilson [...]. This stability theorem is important because many of our computations concerning the Hodge numbers relied on the Hodge decomposition, which is true only in the Kähler case.

5 What is a large complex structure limit?

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