Basics of Seiberg-Witten theory

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The difficulty in summarising Seiberg-Witten theory is that the Seiberg-Witten equations are complicated. The data they involve—Spin^c-structures, complex spinors, coupled Dirac operators—are themselves tricky to understand, never mind the analysis required to understand their solutions. But, while the equations appear awkward, the moduli spaces of Seiberg-Witten solutions modulo gauge are compact, oriented, typically zero-dimensional manifolds, and this makes Seiberg-Witten invariants considerably easier to compute than Donaldson's instanton invariants. Witten's paper of November 1994 [W] suppresses detailed calculations, but the basic arguments are beautifully clear; it is this style, rather than the useful but impenetrable thicket of detail provided by Nicolaescu [N], for instance, that I would like to emulate in this lecture. I have also made considerable use of Donaldson's review [Do]. For an elegant discussion of spinors consult [De].

1 Spin^c-structures

The compact Lie group $\mathrm{Spin}^c(n)$ is defined to be

$$\mathrm{Spin}^c(n) = \frac{\mathrm{Spin}(n) \times \mathrm{U}(1)}{\pm(1,1)} \tag{1}$$

where $\mathrm{Spin}(n)$ is the unique Lie group which double covers $\mathrm{SO}(n)$, realised as the following subset of the Clifford algebra $\mathrm{Cliff}(n)$ associated to the negative of the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^n$:

$$\mathrm{Spin}(n) = \{x_1 \cdots x_{2k} \in \mathrm{Cliff}(n) : \; x_i \in \mathbb{R}^n; \; \|x_i\| = 1\}.$$

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We have

$$\begin{aligned}
\mathrm{Spin}^c(3) &\cong (\mathrm{SU}(2)\times \mathrm{U}(1))/\pm(1,1) \cong \mathrm{U}(2);\\
\mathrm{Spin}^c(4) &\cong (\mathrm{SU}(2)\times \mathrm{SU}(2)\times \mathrm{U}(1))/\pm(1,1,1)\\
&\cong \{(A,B)\in \mathrm{U}(2)\times \mathrm{U}(2) : \det(A)=\det(B)\}.
\end{aligned}$$

A **Spin^c-structure** on an oriented Riemannian n-manifold (X,g) is a lift of the structure group from $\mathrm{SO}(n)$ to $\mathrm{Spin}^c(n)$ under the natural S^1-fibration $\pi:\mathrm{Spin}^c(n)\to \mathrm{SO}(n)$. Thus if $\{U_\alpha\}$ is a cover of X, all of whose multiple intersections are contractible, then a Spin^c-structure is encoded by a Čech 1-cocycle $\tilde g_{\alpha\beta}: U_\alpha\cap U_\beta \to \mathrm{Spin}^c(n)$ such that the maps $g_{\alpha\beta}=\pi\circ\tilde g_{\alpha\beta}$ are the transition functions for (TX,g). The lifted transition functions define a principal bundle $P_{\mathrm{Spin}^c(2n)}$ and a homomorphism of principal bundles $P_{\mathrm{Spin}^c(2n)}\to P_{\mathrm{SO}(2n)}$ which restricts fibrewise to the covering map $\mathrm{Spin}^c(2n)\to \mathrm{SO}(2n)$.

- A **Spin-structure**, that is, a lift of the structure group to $\mathrm{Spin}(n)$, induces a Spin^c-structure via the map

$$\mathrm{Spin}(n)\to \mathrm{Spin}^c(n): \quad x\to \pm(x,1).$$

- An almost complex structure—a lift of the structure group from $\mathrm{SO}(2k)$ to $\mathrm{U}(k)$—also yields a canonical Spin^c-structure, because the natural map $\mathrm{U}(k)\to\mathrm{SO}(2k)$ factors through $\mathrm{Spin}^c(2k)$.

- Hirzebruch and Hopf proved that any smooth oriented manifold of dimension ≤ 4 admits a Spin^c-structure (see [GS]); in contrast, a smooth oriented 4-manifold admits a Spin-structure iff its intersection form is even, iff $w_2=0$.

Let $\mathfrak{s}$ be a Spin^c-structure with 1-cocycle $\tilde g_{\alpha\beta}=(\pm h_{\alpha\beta},\pm\zeta_{\alpha\beta})$. Then $z_{\alpha\beta}:=(\pm\zeta_{\alpha\beta})^2$ defines a $\mathrm{U}(1)$-valued 1-cocycle, and hence a complex line bundle L. We call L the **determinant line bundle** $\det(\mathfrak{s})$ of the Spin^c-structure $\mathfrak{s}$. Write $c_1(\mathfrak{s})$ for its **first Chern class** $c_1(L)\in H^2(X;\mathbb{Z})$, which is represented by the $\mathbb{Z}$-valued 2-cocycle $n_{\alpha\beta\gamma}=\frac{1}{2\pi i}\log(z_{\alpha\beta}z_{\beta\gamma}z_{\beta\gamma})$; it reduces mod 2 to $w_2(X)$.

We regard Spin^c-structures as **equivalent** if their associated 1-cocycles $\tilde g_{\alpha\beta}$ and $\tilde h_{\alpha\beta}$ are cohomologous, in the sense that $\epsilon_\beta\tilde g_{\alpha\beta}=\tilde h_{\alpha\beta}\epsilon_\alpha$ for a 0-cocycle $\epsilon_\alpha: U_\alpha\to(1,\pm1)\subset \mathrm{Spin}^c(n)$, and write $\mathrm{Spin}^c(X)$ for the set of equivalence classes. There is a free, transitive right action of the group $\check H^1(X;S_1)$ on $\mathrm{Spin}^c(X)$. Thus, if $\mathrm{Spin}^c(X)\neq\emptyset$ then we can identify it *non-canonically* with $\check H^1(X;S^1)$, and hence with the group of smooth line bundles. Given

$\mathfrak{s}_1, \mathfrak{s}_2 \in \mathrm{Spin}^c(X)$ we can write $\mathfrak{s}_2 = \mathfrak{s}_1 \otimes L$ for a line bundle L, so that $\det \mathfrak{s}_2 = \det \mathfrak{s}_1 \otimes L^2$ and crucially

$$c_1(\mathfrak{s}_2) = c_1(\mathfrak{s}_1) + 2c_1(L). \tag{2}$$

If $\mathfrak{s}$ is the *canonical* Spin^c-structure of an almost complex structure, then $\det(\mathfrak{s})$ is K_X^{-1}, the dual of the canonical line bundle, and we can write a general Spin^c-structure as $K_X^{-1} \otimes L$.

2 Spinor bundles

A basic fact about real Clifford algebras is that there is a $\mathbb{Z}/2$-graded $\mathbb{C}$-vector space $\mathbb{S}_{2n} = \mathbb{S}_{2n}^+ \oplus \mathbb{S}_{2n}^-$ and an isomorphism of $\mathbb{Z}/2$-graded (or 'super') algebras

$$\mathrm{cl} : \mathrm{Cliff}(2n) \otimes \mathbb{C} \to \mathrm{End}(\mathbb{S}_{2n}).$$

In fact, a choice of almost complex structure J on $V = \mathbb{R}^{2n} \otimes \mathbb{C}$ effects a splitting $V = V^{1,0} \oplus V^{0,1}$ of V into $\pm i$-eigenspaces for J, and we can take

$$\mathbb{S}_{2n} = \Lambda^{0,\bullet} V = \Lambda^\bullet V^{0,1},$$

with the $\mathbb{Z}/2$-grading induced by the natural $\mathbb{Z}$-grading by degree. In particular, $\dim_{\mathbb{C}} \mathbb{S}_{2n}^+ = \dim_{\mathbb{C}} \mathbb{S}_{2n}^- = 2^{n-1}$. Via the inclusion map $i : \mathrm{Spin}(2n) \hookrightarrow \mathrm{Cliff}^+(2n)$ we obtain a representation

$$\rho : \mathrm{Spin}^c(2n) \to \mathrm{Aut}(\mathbb{S}_{2n}) : \quad \pm(x, z) \mapsto z\, \mathrm{cl}(i(x))$$

which decomposes into representations $\rho_\pm : \mathrm{Spin}^c(2n) \to \mathrm{Aut}(\mathbb{S}_{2n}^\pm)$.

Consider an oriented Riemannian $2n$-manifold (X, g) and a Spin^c-structure with principal bundle $P_{\mathrm{Spin}^c(2n)}$. Form the associated bundles

$$\mathbb{S}_{\mathfrak{s}} = P_{\mathrm{Spin}^c(2n)} \times_\rho \mathbb{S}_{2n}, \quad \mathbb{S}_{\mathfrak{s}}^\pm = P_{\mathrm{Spin}^c(2n)} \times_{\rho_\pm} \mathbb{S}_{2n}^\pm.$$

Sections of these bundles are called (positive or negative) **complex spinors**.

On a 4-manifold (X, g) we can be more explicit. Working over a chart U, we have

$$\mathbb{S}_{\mathfrak{s}}^+(U) = \Lambda^0(T^*U) \oplus \Lambda^{0,2}(T^*U), \quad \mathbb{S}_{\mathfrak{s}}^-(U) = \Lambda^{1,0}(T^*U). \tag{3}$$

Note first that

$$\Lambda^2 \mathbb{S}_{\mathfrak{s}}^+ \cong \det(\mathfrak{s}) \cong \Lambda^2 \mathbb{S}_{\mathfrak{s}}^-,$$

an observation which it is enough to verify locally. Next, we can use the metric to identify $\Lambda^{0,2}$ with the bundle Λ^2_+ of complexified self-dual 2-forms:

$$\mathbb{S}_{\mathfrak{s}}^+ = \Lambda^0(T^*U) \oplus \Lambda^2_+(T^*U). \tag{4}$$

Thus locally a positive spinor decomposes into two pieces, a function $X \to \mathbb{C}$ and a complexified self-dual 2-form. In the case of the canonical Spin^c-structure of an almost complex 4-manifold formulae (3) and (3) hold globally, but in general we must twist by sections of a complex line bundle.

3 Dirac operators

Let ∇ denote the Levi-Civita connection on (X^{2n}, g), regarded as a connection on the principal frame bundle $P_{\mathrm{SO}(2n)}$. Given a Spin-structure on X we can identify the Lie algebras of Spin$(2n)$ and SO$(2n)$ and so obtain a connection on $P_{\mathrm{Spin}(2n)}$, hence also on the associated spinor bundles. If we only have a Spin^c-structure $\mathfrak{s}$ then we need a bit more information, since Spin$^c(2n)$ has Lie algebra $\mathfrak{so}(2n) \oplus \mathfrak{u}(1)$. However, if we take in addition a unitary connection A on $\det(\mathfrak{s})$ then we do obtain a connection ∇^A on the complex spinor bundle $\mathbb{S}_{\mathfrak{s}}$, which respects the $\mathbb{Z}/2$-decomposition. Locally this connection looks like

$$\nabla^A = d + +\frac{1}{2}iA + \frac{1}{2}\sum_{i<j} \Omega_{ij} e_i e_j, \tag{5}$$

where the e_i are an orthonormal frame field for TX, and they act by Clifford multiplication, explained in a moment. The **Dirac operator** $D_A : \Gamma(\mathbb{S}_{\mathfrak{s}}) \to \Gamma(\mathbb{S}_{\mathfrak{s}})$ is now defined by

$$D_A s = \sum e_i(\nabla^A_{e_i} s).$$

This needs some thinking through: $\nabla^A s$ lives in $\Gamma(T^*X \otimes \mathbb{S}_{\mathfrak{s}})$, and the covariant derivative $\nabla^A_{e_i} s$ contracts this to a section of $\mathbb{S}_{\mathfrak{s}}$. Now over each point $x \in X$, e_i can be considered as an element of Cliff$(2n)$, hence as an endomorphism of $\mathbb{S}$; thus $e_i(\nabla^A_{e_i} u) \in \Gamma(\mathbb{S}_{\mathfrak{s}})$. The jargon is that e_i acts by **Clifford multiplication** $\mathrm{cl} : TM \otimes \mathbb{S}_{\mathfrak{s}} \to \mathbb{S}_{\mathfrak{s}}$. A coordinate-free expression for D_A is

$$\begin{aligned} D_A = \mathrm{cl} \circ \nabla_A :\ \Gamma(\mathbb{S}_{\mathfrak{s}}) &\xrightarrow{\nabla_A} \Gamma(T^*M \otimes \mathbb{S}_{\mathfrak{s}}) \\ &\xrightarrow{\text{metric}} \Gamma(TM \otimes \mathbb{S}_{\mathfrak{s}}) \\ &\xrightarrow{\mathrm{cl}} \Gamma(\mathbb{S}_{\mathfrak{s}}). \end{aligned}$$

The Dirac operator satisfies a Weitzenböck identity due to Lichnierowicz ('one of the most fruitful calculations in differential geometry'—SKD):

$$D_A^* D_A = (\nabla^A)^* \nabla^A + \frac{s}{4} + \frac{1}{2}\mathrm{cl}(F_A^+), \tag{6}$$

where s is the scalar curvature of g, and the $i\mathbb{R}$-valued curvature 2-form F_A can act by Clifford multiplication by identifying $\frac{1}{2}x \wedge y$ with the Clifford product xy. One can check that D_A exchanges sections of the subbundles $\mathbb{S}^{\pm}_{\mathfrak{s}}$; in fact, usually the notation D_A is used for the map $\Gamma(\mathbb{S}^+_{\mathfrak{s}}) \to \Gamma(\mathbb{S}^-_{\mathfrak{s}})$. I remark without elaboration that D_A is a generalised Laplacian, that is its symbol map (re)defines a Clifford algebra stucture.

4 The Seiberg-Witten equations

Fix the following data:

- A smooth, oriented[1], Riemannian 4-manifold (X, g); we won't assume compactness yet.
- A Spin^c-stucture $\mathfrak{s}$ on X, with determinant line bundle $L = \det(\mathfrak{s})$.
- A hermitian metric h on L.
- A closed, imaginary 2-form η, the **perturbation parameter**.

The Spin^c-structure $\mathfrak{s}$ defines complex spinor bundles $\mathbb{S}^{\pm}_{\mathfrak{s}}$. Define the **configuration space** $\mathcal{A}_{\mathfrak{s}}$ to be the set of pairs (ψ, A) where

- ψ is a section of $\mathbb{S}^+_{\mathfrak{s}}$ (a 'spinor field' on X);
- A is a unitary connection on L (a 'potential').

For each such A there is a Dirac operator $D_A : \Gamma(\mathbb{S}^+_{\mathfrak{s}}) \to \Gamma(\mathbb{S}^-_{\mathfrak{s}})$. The **Seiberg-Witten equations** are:

$$\boxed{\begin{aligned} &D_A\psi = 0 && \text{(Dirac equation)} \\ &\mathrm{cl}(F_A^+ + \eta^+) = \frac{1}{2}q(\psi) && \text{(monopole equation)} \end{aligned}}$$

The Dirac equation takes place in $\Gamma(\mathbb{S}^-_{\mathfrak{s}})$, the monopole equation in $\mathrm{End}(\mathbb{S}^+_{\mathfrak{s}})$. Here $q(\psi) \in \mathrm{End}(\mathbb{S}^+_{\mathfrak{s}})$ is the traceless symmetric endomorphism

$$q(\psi) = \overline{\psi} \otimes \psi - \frac{1}{2}|\psi|^2\mathrm{Id},$$

or in matrix form

$$q\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \overline{\alpha}\beta \\ \overline{\beta}\alpha & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{pmatrix}.$$

[1] I mean orient*ed*, not just orient*able*, i.e. we fix an isomorphism $H^4(X; \mathbb{Z}) \to \mathbb{Z}$.

4.1 Gauge symmetry

The Seiberg-Witten equations have more symmetry than meets the eye: they are preserved by a (left) action of the U(1) **gauge group** $\mathcal{G} = \mathrm{Map}(X, \mathrm{U}(1))$. Let's try to understand how this group acts.

First, U(1) acts by scalar multiplication on the vector spaces $\mathbb{S}_4^{\pm}$, so (from their definition as associated bundles) we get a fibre-preserving action of $\mathcal{G}$ on $\mathbb{S}_{\mathfrak{s}}^{\pm}$. Hence $\mathcal{G}$ acts on sections $\Gamma(\mathbb{S}_{\mathfrak{s}}^{\pm})$. Now things get more subtle: $\mathcal{G}$ naturally acts by conjugation on the set of connections on $\mathbb{S}_{\mathfrak{s}}$; given our unitary connection A on L, we obtain a covariant derivative ∇^A on $\mathbb{S}_{\mathfrak{s}}$, and $\gamma \cdot \nabla^A := \gamma \nabla^A \gamma^{-1}$. But this connection is also induced from a unitary connection on L, namely $A - 2(d\gamma)\gamma^{-1}$. The factor of 2 looks strange if you're used to Yang-Mills theory—to see where it comes from look at equation 5. Finally, we have a formula for the action on configuration space $\mathcal{A}_{\mathfrak{s}}$:

$$\gamma \cdot (\psi, A) = (\gamma\psi, A - 2d\gamma.\gamma^{-1});$$

using this formula you can deduce the invariance of the Seiberg-Witten equations. The **Seiberg-Witten moduli space** associated to our data is defined to be

$$\mathcal{M}_{\mathfrak{s}} = \{\text{Seiberg-Witten solutions}\}/\mathcal{G} \subset \mathcal{A}_{\mathfrak{s}}/\mathcal{G}.$$

4.2 Involution

There is an additional symmetry of the equations, this time beween solutions with perturbation parameter η and those with parameter $-\eta$. It is induced by the involution $(h_{\alpha\beta}, z_{\alpha\beta}) \mapsto (h_{\alpha\beta}, \overline{z_{\alpha\beta}})$ on $\mathrm{Spin}^c(X)$.

5 Properties of the equations

Suppose (ψ, A) is an L^2-solution to the Seiberg-Witten equations on X (e.g. any solution when X is compact). By the Weitzenböck formula (6),

$$0 = D_A^* D_A \psi = (\nabla^A)^* \nabla^A \psi + \frac{s}{4}\psi + \frac{1}{2} F_A^+(\psi)$$

(recall s is scalar curvature). Taking inner products with ψ, integrating, and simplifying using the monopole equation, it's straightforward to show

$$\int_X \left(|\nabla^A \psi|^2 + \frac{1}{4}|\psi|^4 - (\eta^+(\psi), \psi) \right) \mathrm{vol}_g = -\int_X \frac{s}{4}|\psi|^2 \mathrm{vol}_g. \qquad (7)$$

5.1 Non-negative scalar curvature

Suppose $s \geq 0$ everywhere, and take $\eta = 0$. Then the LHS of (7) is non-negative while the RHS is non-positive, hence both are zero. So $\psi \equiv 0$ and $F_A^+ \equiv 0$. We will discuss solutions of this form later; the point is that, when $b_2^+ > 0$, for generic metrics they do not exist. We will conclude from this that on a closed 4-manifold with $b_2^+ > 0$ admitting a metric of non-negative scalar curvature the Seiberg-Witten invariants vanish identically. Both this statement and its contrapositive are useful in 4-manifold topology.

5.2 Finiteness

Now suppose $s_0 := -\min_X(s) \geq 0$ on the compact manifold X. Again, take a solution (ψ, A) to the $\eta = 0$ equations. By (7) we have

$$\int_X (|\nabla^A \psi|^2 + \frac{1}{4}|\psi|^4)\mathrm{vol}_g \leq \frac{s_0}{4}\int_X |\psi|^2 \mathrm{vol}_g \leq \frac{s_0}{4}\sqrt{\mathrm{Vol}(X)\int_X |\psi^4|\mathrm{vol}_g}$$

where the second inequality is an application of Cauchy-Schwartz. Throw away the $|\nabla^A \psi|^2$ term and rearrange, to obtain the bound

$$\int_X |\psi|^4 \leq s_0^2 \mathrm{Vol}(X).$$

This in turn yields an L^2 upper bound on F_A^+, and since by Chern-Weil theory we have

$$c_1^2(\mathfrak{s})[X] = \frac{1}{4\pi^2}\int_X F_A^2 = \frac{1}{4\pi^2}\int_X (|F_A^+|^2 - |F_A^-|^2)\mathrm{vol}_g,$$

$c_1^2(\mathfrak{s})[X]$ is bounded above. We will see later from an index calculation (8) that for generic metrics there are no solutions when $c_1^2(\mathfrak{s})[X] < 2\chi(X) + 3\tau(X)$, where χ and τ are Euler characteristic and signature. So $c_1^2(\mathfrak{s})$ is bounded in $H^4(X;\mathbb{Z})$, hence $c_1(\mathfrak{s})$ is bounded in $H^2(X;\mathbb{Z})$, hence there are only finitely many Spin^c-structures on X for which the $\eta = 0$ Seiberg-Witten equations admit solutions.

6 The gauge theory strategy

In gauge theory one considers the space $\mathcal{M}_{X,E}(g)$ of solutions modulo gauge transformations to a PDE on a vector bundle E over a smooth oriented manifold X. Solutions to the PDE are invariant under a group of bundle automorphisms, the **gauge group**. We are interested just in the orientation-preserving diffeomorphism type of X; however, the PDE will involve some auxiliary data g (metrics on X and E, perturbation parameters, ...) which are not preserved by orientation-preserving diffeomorphisms. The idea, then, is this:

- If the gauge group acts freely, the configuration space divided by the gauge group action will be a smooth Banach manifold, containing the moduli space $\mathcal{M}(g)$. There may be **reducible solutions**, which have nontrivial stabilizer. We must arrange that for generic[2] data g these reducibles do not hit $\mathcal{M}(g)$. More than this, we need to show that reducibles do not occur in **generic 1-parameter families** of data g. This will allow us to construct 'cobordisms' between moduli spaces $\mathcal{M}(g_1)$ and $\mathcal{M}(g_2)$ in such a way as to show that $\mathcal{M}(g)$ does not depend on g.

- If the equations are **elliptic** then away from reducibles their linearizations will define Fredholm maps between Banach manifolds. For generic choices of the data g, the moduli space $\mathcal{M}(g)$ will then be a smooth finite-dimensional manifold (Sard-Smale theorem); reducible solutions would introduce singularities. The dimension of $\mathcal{M}(g)$, for generic g, is the **index** of the equations; this can be computed using the Atiyah-Singer index theorem.

- In some cases the moduli spaces will be compact; if not, one must construct some appropriate compactification, typically as a completion in the ambient space. The 'ends' of the compactification often have a natural geometric description.

- One must define an orientation for the moduli space. Writing T for the linearization of the operators defining the PDE, $\ker(T) \oplus (\operatorname{coker}(T))^*$ defines an element of the real K-theory on X, the **index bundle** $\operatorname{ind}(T)$; its top exterior power is actually a real line bundle $\det \operatorname{ind}(T)$, and Donaldson gave a recipe for producing an orientation of the moduli space from an orientation of $\det \operatorname{ind}(T)$.

This list is to some extent an idealisation; one must consider variants, such as isolated reducibles appearing in generic 1-parameter families; and to get the arguments to work one must usually allow solutions in certain Sobolev spaces, then use a 'bootstrapping' argument to show that this doesn't affect the

[2] A property holds generically in a metric space if it holds on a countable intersection of open, dense subsets, itself dense by Baire's theorem.

topology of the moduli space. However, it does represent the basic strategy used in a number of problems—one of course, being Seiberg-Witten theory:

- Instantons (Donaldson theory);
- Pseudo-holomorphic curves in symplectic manifolds (Gromov, Gromov-Witten invariants);
- Floer homology (Lagrangian, instanton, Seiberg-Witten, etc.) and 3-manifold invariants (Casson, etc.).

The moduli spaces in Donaldson or Seiberg-Witten theory can be used in two basic ways. First, to give non-existence proofs, e.g. for smooth, closed, oriented 4-manifolds with negative definite, non-diagonal intersection form, by deriving a contradiction from the topology of the moduli space. Second, to define invariants of such manifolds. The strategy is then

- If the moduli space $\mathcal{M}$ is a 0-dimensional compact manifold then it is a finite set; one counts the points with signs obtained by comparing the canonical orientation of a finite set with the orientation constructed on $\mathcal{M}$, and obtains an invariant

$$\sum_{x\in\mathcal{M}} \varepsilon_x, \quad \varepsilon_x = \pm 1.$$

- In higher dimensions, construct a map $\mu : H_2(X;\mathbb{Z}) \to H^2(\mathcal{M};\mathbb{Z})$. If $\mathcal{M}$ has even dimension $2d$ then define invariants as

$$\langle \mu([\Sigma_1]) \cup \cdots \cup \mu([\Sigma_d]), [\mathcal{M}]\rangle.$$

  We will only discuss a cheap version of this, using a fixed class $U \in H^2(\mathcal{M};\mathbb{Z})$.

7 The Seiberg-Witten case

Reducibles. If the gauge group $\mathcal{G}$ fails to act freely on (ψ, A) then clearly $\psi = 0$; then $F_A^+ + \eta^+ = 0$, and the stabilizer of (ψ, A) is $\mathrm{U}(1)$. Consider the unperturbed case $\eta = 0$ (so we have an 'abelian instanton'). Now $F_A/2\pi i$ represnts $c_1(L)$, so it lies in the integral lattice in $H^2(X;\mathbb{Z})$, and $F_A^+ = 0$, so it lies in the anti-self-dual subspace. But if $b_2^+ \geq 1$ then for generic metrics an integral, anti-self-dual harmonic form is zero, and for $b_2^+ \geq 2$ the same is true for generic 1-parameter families of metrics. So to get absolute invariants we must assume $b_2^+ \geq 2$, while for $b_2^+ = 1$ we will get 'chambers' in the space of metrics, giving different invariants. Note how useful it was that we considered pairs (ψ, A), not just connections A.

Index. Witten shows that the linearization of the Seiberg-Witten equations is elliptic, of index

$$\mathrm{ind}(d^* \oplus d^+) + \mathrm{ind}(D_A),$$

where both terms are given by standard formulae. The result is that the 'expected' dimension of $\mathcal{M}_{\mathfrak{s}}$ is

$$\frac{1}{4}\{c_1(L)^2 - (2\chi(X) + 3\tau(X)\} \tag{8}$$

where χ and τ denote Euler characteristic and signature. If the index is negative the moduli space will generically be empty. One feature of this formula is immediately striking: on an almost complex 4-manifold, $c_1^2(TX)[X] = 2\chi + 3\tau$, so conveniently the expected dimension for the canonical Spin^c-structure is zero.

Compactness. The Seiberg-Witten moduli spaces are always compact, unlike the instanton moduli spaces. Witten remarks that the lack of compactness in the instanton theory can be explained in the following way: the equations are conformally invariant; there exists a nontrivial L^2 instanton on flat $\mathbb{R}^4$; such a solution may be embedded into a small region of any 4-manifold, giving a highly localised approximate solution; this may sometimes be perturbed to an exact solution. There may therefore be a sequence of instanton solutions shrinking to zero size, and such a sequence has no convergent subsequence (think of approximations to the delta distribution). In contrast, we saw in out discussion of non-negative scalar curvature that the Seiberg-Witten equations admit no nontrivial L^2 solutions on flat $\mathbb{R}^4$. One can prove compactness by establishing an L^∞ bound on the curvature F_A using the maximum principle, then applying elliptic theory.

Orientation. In Seiberg-Witten theory the orientation on $\det \mathrm{ind}(T)$ arisies from an orientation of the vector space $H^2_+(X;\mathbb{R}) \oplus H^1(X;\mathbb{R})$.

7.1 Invariants

In the 0-dimensional case we make a signed count of solutions, setting

$$\mathbf{SW}_X(\mathfrak{s}) = \sum_{x \in \mathcal{M}} \varepsilon_x, \quad \varepsilon_x = \pm 1.$$

In higher dimensions we can use the pullback square

$$\begin{array}{ccc} i^*\mathbb{U}_{\mathfrak{s}} & \longrightarrow & \mathbb{U}_{\mathfrak{s}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{M}_{\mathfrak{s}} & \xrightarrow{i} & \mathcal{A}_{\mathfrak{s}}/\mathcal{G} \end{array}$$

where $\mathbb{U}_{\mathfrak{s}} = \mathcal{A}_{\mathfrak{s}}/\mathrm{Stab}_{\mathcal{G}}(\mathrm{pt})$. Since $\mathbb{U}_{\mathfrak{s}} \to \mathcal{A}_{\mathfrak{s}}/\mathcal{G}$ is a line bundle away from reducibles, we can take first Chern classes and so obtain an element $U = c_1(i^*\mathbb{U}_{\mathfrak{s}}) \in H^2(\mathcal{M}_{\mathfrak{s}};\mathbb{Z})$. Now define

$$\mathbf{SW}_X(\mathfrak{s}) = \langle (1-U)^{-1}, [\mathcal{M}_{\mathfrak{s}}]\rangle.$$

Note that this can be nonzero only when $\dim \mathcal{M}_{\mathfrak{s}}$ is even. When $b_2^+ > 1$ $\mathbf{SW}_X(\mathfrak{s})$ is an invariant; when $b_2^+ = 1$ it takes two values, depending on η and g, and we may consider it crudely as an unordered pair of integers.

A less sensitive but more readily computable invariant is

$$\mathrm{Bas}(X) = \{c_1(\det(\mathfrak{s})) : \mathfrak{s} \in \mathrm{Spin}^c(X);\ \mathbf{SW}_X(\mathfrak{s}) \neq 0\} \subset H^2(X;\mathbb{Z}),$$

the set of Seiberg-Witten **basic classes**. In all known cases where $b_2^+ > 1$, the basic classes correspond to 0-dimensional moduli spaces, but when $b_2^+ = 1$ this is frequently not the case.

8 Seiberg-Witten invariants of Kähler surfaces: Witten's factorization method

In this section (X,ω) is a compact Kähler surface with canonical line bundle K_X. We'll also assume $b_2^+ > 1$ with respect to the canonical orientation $\omega^2 > 0$: since $b_2^+ = 2p_g + 1$ we can then take as perturbation parameter a **holomorphic 2-form** $\eta \in H^0(X;K_X)$. As we've seen, the canonical Spin^c-structure $\mathfrak{s}_0$ has determinant line bundle K_X^{-1}, and spinor bundles

$$\mathbb{S}_0^+ = \Lambda^{0,0}T^*X \oplus \Lambda^{0,2}T^*X = \underline{\mathbb{C}} \oplus K_X^{-1}, \quad \mathbb{S}_0^- = \Lambda^{0,1}T^*X.$$

A general Spin^c-structure can now be obtained by twisting: $\mathfrak{s} = \mathfrak{s}_0 \otimes L$; $\det(\mathfrak{s}) = K_X^{-1} \otimes L^2$. The spinor bundles $\mathbb{S}_L^\pm := \mathbb{S}_{\mathfrak{s}}^\pm$ are now

$$\mathbb{S}_L^+ = L \oplus (L \otimes K_X^{-1}), \quad \mathbb{S}_L^- = L \otimes \Lambda^{0,1}T^*X.$$

Using this splitting we write an even spinor $\psi \in \Gamma(\mathbb{S}_L^+)$ as

$$\psi = \alpha + \beta, \quad \alpha \in \Gamma(L),\ \beta \in \Gamma(L \otimes K_X^{-1}) = \Gamma(L \otimes \Lambda^{0,2}).$$

The space of complexified self-dual 2-forms can be identified with $\Lambda^{2,0} \oplus \mathbb{C}.\omega \oplus \Lambda^{0,2}$, and using this decomposition Witten rewrites the η-perturbed monopole equation as

$$F^{2,0} = \overline{F^{0,2}} = \overline{\alpha}\beta - \eta, \quad iF^{1,1} \cdot \omega = |\beta|^2 - |\alpha|^2,$$

where $F = F_B$ is the curvature of a connection B on L. The Dirac operator D_A can be written in the form $\sqrt{2}(\overline{\partial}_B + \overline{\partial}_B^*)$. The coupled Dolbeault operator $\overline{\partial}_B$ is a map $\Gamma(L) \to \Gamma(L \otimes \Lambda^{0,1})$—it kills $\Gamma(L \otimes \Lambda^{0,2})$—while its dual $\overline{\partial}_B^*$ is a map $\Gamma(L \otimes \Lambda^{0,2}) \to \Gamma(L \otimes \Lambda^{0,1})$. So the Dirac equation is

$$\overline{\partial}_B \alpha + \overline{\partial}_B^* \overline{\beta} = 0.$$

Therefore

$$F^{0,2}\alpha = \overline{\partial}_B \overline{\partial}_B \alpha = -\overline{\partial}_B \overline{\partial}_B^* \overline{\beta},$$

so when $\eta = 0$ we have $\overline{\partial}_B \overline{\partial}_B^* \overline{\beta} + |\alpha|^2 \beta = 0$ and

$$\int_X (|\overline{\partial}_B^*|^2 + |\alpha|^2 |\beta|^2) \mathrm{vol}_g.$$

Hence $F^{0,2} = \overline{\alpha}\beta = 0$. By the Newlander-Nirenberg theorem, $\overline{\partial}_B$ defines a holomorphic structure on L. In particular, $c_1(L)$ is of type $(1,1)$.

We now drop the assumption $\eta = 0$, but we *can* assume $\int_X F \wedge \eta = \int_X F^{0,2} \wedge \eta = 0$, because basic classes must have type $(1,1)$. A similar procedure now shows

$$\int_X (|\overline{\partial}_B^*|^2 + |\alpha\overline{\beta} - \eta|^2) \mathrm{vol}_g = 0.$$

So we find:

- $F^{0,2} = 0$. Thus B defines a holomorphic structure on L.
- $\overline{\partial}_B \alpha = 0$, so α is a holomorphic section of L.
- $\overline{\partial}_B^* \beta = 0$, so $\overline{\beta}$ is a holomorphic section of $L^* \otimes K_X$.
- $\alpha\overline{\beta} = \eta$: solutions describe a **holomorphic factorization** of the perturbation parameter.

Conversely, given such data we can go backwards and find a unique solution to the equations. So the slogan is

Everything reduces to algebraic geometry!

A consequence of this factorization is that solutions of the monopole equation are *isolated*—think of their divisors—and so the moduli space is at most

0-dimensional. From the dimension formula, we find that a basic class x satisfies

$$x^2 = K_X^2.$$

Of course, a Spin^c-structure with non-empty 0-dimensional moduli space may not define a basic class: the signs attached to the points of the moduli space could cancel out. Writing the Poincaré dual of η as $\sum r_i[C_i]$ for disjoint connected curves C_i and $r_i \geq 0$, we can see from the factorization that if the line bundle L corresponds to a basic class then the divisor of a section of L has the form $\sum s_i[C_i]$ with $0 \leq s_i \leq r_i$. Integrating the Kähler form ω on the C_i we see that if $\kappa \in \mathrm{Bas}(X)$ then

$$0 \leq |\kappa \cdot [\omega]| = \left| \sum (2s_i - r_i) \int_{C_i} \omega \right| \leq \sum r_i \int_{C_i} \omega = K \cdot \omega.$$

Equality occurs iff all $s_i = 0$ or $s_i = r_i$, which means either L or $K_X \otimes L^*$ is trivial, i.e. $\kappa = K_X^{\pm 1}$. In either case there is a single solution, so (conflating line bundles with Spin^c-structures) we have $\mathbf{SW}_X(K_X^{\pm 1}) = \pm 1$.

To summarise:

(X, ω) Kähler, $p_g(X) > 0$, $\kappa \in \mathrm{Bas}(X)$. Then $\kappa^2 = K_X^2$ and

$\kappa \in H^2(X; \mathbb{Z}) \cap H^{1,1}(X)$;	$0 \leq \lvert\kappa \cdot \omega\rvert \leq K_X \cdot \omega$;
$\lvert\kappa \cdot \omega\rvert = K_X \cdot \omega \Leftrightarrow \kappa = \pm K_X$;	$\mathbf{SW}_X(K_X^{\pm 1}) = \pm 1$.

An immediate consequence is that if K_X is torsion (which is true for tori, K3s, Enriques and hyperelliptic surfaces) we have

$$\mathrm{Bas}(X) = \{\pm K_X\}.$$

The same is true for minimal surfaces with K_X ample ('general type'): you can prove this in a couple of lines using the Hodge index theorem and the fact $K^2 > 0$. Consequently orientation-preserving diffeomorphisms between manifolds of these types preserve the canonical class, up to sign[3]. On minimal elliptic surfaces the set of basic classes is larger, and distinguishes various diffeomorphism types. If $\tilde{X}$ is a blow-up of X at a point, with exceptional divisor Poincaré dual to E, then

$$\mathrm{Bas}(\tilde{X}) = \text{proper transform}(\mathrm{Bas}(X)) \pm E.$$

[3]Take care: an orientation-preserving diffeomorphisms $f : X \to Y$ induces $f^* : \mathrm{Spin}^c(Y) \to \mathrm{Spin}^c(X)$, and $\mathbf{SW}_Y = \pm \mathbf{SW}_X \circ f^*$. However, in general f^* will not be compatible with our identification of Spin^c-structures with cohomology classes.

A final point: we used the canonical orientation on X, determined by setting $\omega^2 > 0$. However, if b_2^+ and b_2^- were both > 1 we could have used the opposite orientation. We would have found that the Seiberg-Witten invariants all vanished. It seems to be typical of smooth orientable 4-manifolds that the Seiberg-Witten invariants are nontrivial in at most one orientation.

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